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A note on equitable colorings of forests*

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Abstract

This note gives a short proof on characterizations of a forest to be equitably k-colorable.

1 Introduction

In a graph $G = (V, E)$, a *stable set* (or *independent set*) is a pairwise non-adjacent vertex subset of V. The *stability number* (or *independence number*) $\alpha(G)$ of G is the maximum size of a stable set in G. An *equitable k-coloring* of $G = (V, E)$ is a partition of V into k pairwise disjoint stable sets C_1, C_2, $\ldots$, C_k such that $||C_i| - |C_j|| \leq 1$ for all i and j. The *equitable chromatic number* $\chi_=(G)$ of G is the minimum number k for which G has an equitable k-coloring.

The notion of equitable colorability was introduced by Meyer [6], who also conjectured a statement stronger than Brooks' theorem that $\chi_=(G) \leq \Delta(G)$ for any connected graph G other than a complete graph or an odd cycle, where $\Delta(G)$ is the maximum degree of a vertex in G. Hajnál and Szemerédi [4] gave a deep result that any graph G is equitably k-colorable for $k > \Delta(G)$. This topic is then studied for many researchers. Lih [5] gave a survey on this line.

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The main concern of this note is on the equitable colorability of trees. Meyer in his paper [6] also showed that a tree T is equitably $(\lceil\frac{\Delta(T)}{2}\rceil+1)$-colorable. However, this proof was faulty. It was reported by Guy [3] that Eggleton remedied the defects. He could prove that a tree T is equitably k-colorable if $k \geq \lceil\frac{\Delta(T)}{2}\rceil+1$. Meyer's results on trees was greatly improved by Bollobás and Guy [1] as follows.

Theorem 1 (Bollobás and Guy [1]) *A tree T of order n is equitably 3-colorable if $n \geq 3\Delta(T)-8$ or $n = 3\Delta(T)-10$.*

Using this result as the induction basis, Chen and Lih [2] gave a complete characterization for a tree to be equitably k-colorable. Their results are in two parts. Notice that as a tree is a connected bipartite graph, its vertex set has a bipartition.

Theorem 2 (Chen and Lih [2]) *Suppose T is a tree of order n, and (A,B) is a bipartition of T. For $||A|-|B|| \leq 1$, the tree T is equitably k-colorable if and only if $k \geq 2$.*

To see their second result, we need another notion. Suppose x is a vertex in a graph $G=(V,E)$. An *x-stable set* in G is a stable set which contains x. The *x-stability number* $\alpha_x(G)$ of the graph G is the maximum size of a x-stable set in G. We use α_x for $\alpha_x(G)$ when there is no ambiguity on the graph G.

Suppose x is a vertex in a graph $G=(V,E)$ of order n. Partition V into $k=\chi_=(G)$ stable sets C_1, C_2, …, C_k such that $||C_i|-|C_j|| \leq 1$ for all i and j. Suppose $x \in C_i$. Then $|C_i| \leq \alpha_x$ and $|C_j| \leq \alpha_x+1$ for all $j \neq i$. Consequently,

$$n = \sum_{i=1}^{k} |C_i| \leq \alpha_x + (k-1)(\alpha_x+1) = \chi_=(G)(\alpha_x+1)-1$$

and so $\chi_=(G) \geq \frac{n+1}{\alpha_x+1}$, which gives (see [2])

$$\chi_=(G) \geq \max_{x\in V}\lceil\tfrac{n+1}{\alpha_x+1}\rceil. \tag{1}$$

Theorem 3 (Chen and Lih [2]) *Suppose T is a tree of order $n \geq 2$, and (A,B) is a bipartition of T. For $||A|-|B|| \geq 2$, the tree T is equitably k-colorable if and only if $k \geq \max\{3, \lceil\frac{n+1}{\alpha_v+1}\rceil\}$, where v is an arbitrary vertex of degree $\Delta(T)$.*

Notice that when $||A|-|B|| \leq 1$, it is the case that $\alpha_x \geq \frac{n-1}{2}$ and so $\frac{n+1}{\alpha_x+1} \leq 2$ for any vertex x. On the other hand, even when $||A|-|B|| \geq 2$, it is still possible that $\frac{n+1}{\alpha_x+1} \leq 2$ for all vertices x. An easy example is the tree obtained from a 3-path by adding $\ell \geq 3$ leaves joining to each vertex of the 3-path. This shows that the 3 in the lower bound of Theorem 3 can not be dropped.

An unpublished manuscript by Miyata, Tokunaga and Kaneko [7] gave another characterization of equitable colorability of trees. While the proof is long, it is without using other results.

Theorem 4 (Miyata, Tokunaga and Kaneko [7]) *Suppose $T = (V, E)$ is a tree of order n and $k \geq 3$ is an integer. Then T is equitably k-colorable if and only if $\alpha_x \geq \lfloor \frac{n}{k} \rfloor$ for any vertex x or equivalently $k \geq \max_{x \in V} \lceil \frac{n+1}{\alpha_x+1} \rceil$.*

Notice that the equivalence follows from that

$$k \geq \lceil \tfrac{n+1}{\alpha_x+1} \rceil \iff k \geq \tfrac{n+1}{\alpha_x+1} \iff \alpha_x \geq \tfrac{n+1}{k} - 1 = \tfrac{n-k+1}{k} \iff \alpha_x \geq \lfloor \tfrac{n}{k} \rfloor.$$

The purpose of this note is to clarify the relation between Theorems 3 and 4. We also give a short proof of the result by combining all techniques in [1, 2, 7] together. We present the proof in terms of forests as it is the same as that for trees.

2 Equitable coloring on forests

We first clarify the relation between taking maximum over all vertices in Theorem 4 and using only one vertex in Theorem 3.

In a graph G, the *neighborhood* $N(v)$ of a vertex v is the set of all vertices adjacent to v, and the *closed neighborhood* $N[v]$ is $\{v\} \cup N(v)$. For a subset S of vertices, the neighborhood $N(S)$ of S is $\cup_{v \in S} N(v)$.

Lemma 5 *Suppose v is a vertex in a forest $F = (V, E)$ of order n. If $\lceil \frac{n+1}{\alpha_v+1} \rceil > 3$, then v is the only vertex of degree $\Delta(F)$. Consequently, if $\max\{3, \max_{x \in V} \lceil \frac{n+1}{\alpha_x+1} \rceil\} > 3$, then the maximum is attained by the unique vertex of degree $\Delta(F)$.*

Proof. Notice that $\lceil \frac{n+1}{\alpha_v+1} \rceil > 3$ implies $\frac{n}{\alpha_v+1} \geq 3$ or $n \geq 3\alpha_v + 3$. Suppose v is of degree d. First, $\alpha_v = 1 + \alpha(F - N[v])$. Notice that the stability number of any bipartite graph is at least the half of its order as the larger part in a bipartition is a stable set. It is then the case that $2\alpha_v = 2 + 2\alpha(F - N[v]) \geq 2 + n - 1 - d \geq 2 + 3\alpha_v + 3 - 1 - d$ and so $\deg(v) = d \geq \alpha_v + 4$. On the other hand, suppose x is a vertex other than v. Then all of its neighbors, except possibly one, form a stable set in $F - N[v]$ since F has no cycles. Hence, $\alpha(F - N[v]) \geq \deg(x) - 1$ and so $\alpha_v = 1 + \alpha(F - N[v]) \geq \deg(x)$, which in turn implies $\deg(v) > \deg(x)$. ∎

Lemma 5 implies that the conditions in Theorems 3 and 4 are in fact the same. Having this in mind, we are ready to re-prove the main assertion.

Theorem 6 *Suppose F is a forest of order n and $k \geq 3$ is an integer. Then F is equitably k-colorable if and only if $\alpha_x \geq \lfloor \frac{n}{k} \rfloor$ for any vertex x.*

Proof. We only prove the sufficiency. Suppose (A,B) is a bipartition of $F=(V,E)$ with $|A|=a\geq|B|=b$. Then $n=a+b$. Without loss of generality, we may assume that A has as few isolated vertices as possible. Let $s_i=\lfloor\frac{n+i-1}{k}\rfloor$ for $1\leq i\leq k$. We only need to partition V into stable sets of size s_1, s_2, …, s_k, respectively. Choose the minimum index j for which $b\leq\sum_{i=1}^{j}s_i$. If the inequality is an equality, we can partition V into desired stable sets. So, we now assume that $\sum_{i=1}^{j-1}s_i<b<\sum_{i=1}^{j}s_i$.

Case 1. $1<j$.

Let S be the set of $s=b-\sum_{i=1}^{j-1}s_i$ vertices of lowest degrees in B. The number of edges between S and A is then at most s times the average degree of a vertex in B, which is at most $\frac{n-1}{b}$. Therefore, $|N(S)|\leq\frac{s(n-1)}{b}<\frac{sn}{b}$ and then

$$|S\cup(A-N(S))|>s+a-\tfrac{sn}{b}=\tfrac{(b-s)a}{b}\geq s_1,$$

since $b-s\geq s_1$ and $a\geq b$. Hence, $|S\cup(A-N(S))|\geq s_1+1\geq s_j$ and we can find a subset S' of A such that $S\cup S'$ is a stable set of size s_j. In this case, the other vertices can be properly partitioned to get an equitable k-coloring of F.

Case 2. $j=1$, i.e., $b<\lfloor\frac{n}{k}\rfloor$.

In this case, by the choice of (A,B), we know that A has no isolated vertices. Denote L the set of all leaves in A. Then, $|L|+2|A-L|\leq\sum_{x\in A}\deg(x)\leq n-1$ and so $|L|\geq |L|+|L|+2|A-L|-(n-1)=2a-n+1=a-b+1$.

We first choose a subset S of B such that the stable set $(N(S)\cap L)\cup(B-S)$ has size at least $\lceil\frac{n}{k}\rceil$. Notice that since $k\geq 3$ and $b<\lfloor\frac{n}{k}\rfloor$, we have $|L|\geq\lceil\frac{n}{k}\rceil$. Hence, B is such a candidate, while $\emptyset$ is not. We may assume that S is chosen so that $|S|$ is smallest. Choose a vertex v from S. Then $|N(S-\{v\})\cap L|+|B-(S-\{v\})|\leq\lceil\frac{n}{k}\rceil-1$.

If the stable set $(N(B-S)\cap L)\cup S$ has size at least $\lfloor\frac{n}{k}\rfloor$, then A has two disjoint subsets S' and S'' such that $S'\cup(B-S)$ and $S''\cup S$ are two stable sets of size s_k and s_1, respectively. Hence the other vertices can be properly partitioned to get an equitable k-coloring of F. So, we may assume that $|N(B-S)\cap L|+|S|\leq\lfloor\frac{n}{k}\rfloor-1$. Adding the two inequalities gives $|L|-|N(v)\cap L|+b+1\leq\lceil\frac{n}{k}\rceil+\lfloor\frac{n}{k}\rfloor-2$. Consequently,

$$|N(v)\cap L|\geq|L|+b+1-\lceil\tfrac{n}{k}\rceil-\lfloor\tfrac{n}{k}\rfloor+2\geq a+4-\lceil\tfrac{n}{k}\rceil-\lfloor\tfrac{n}{k}\rfloor.$$

Since $\alpha_v\geq\lfloor\frac{n}{k}\rfloor$, there is a v-stable set R of size $\lfloor\frac{n}{k}\rfloor$. We may assume that R is chosen so that $|R\cap B|$ is minimum. If $R\cap B=\{v\}$, then as $|(N(v)\cap L)\cup(B-\{v\})|\geq a+4-\lceil\frac{n}{k}\rceil-\lfloor\frac{n}{k}\rfloor+b-1\geq\lceil\frac{n}{k}\rceil$ we can choose a subset S' of A such that $S'\cup(B-\{v\})$ is a stable set of size $\lceil\frac{n}{k}\rceil$. This and R together with a proper partition of other vertices give an equitable k-coloring of F. Suppose $R\cap B$ has at least two vertices. In this case, any vertex $x\in L$ that is not in R must be adjacent to some vertex in $R\cap B$, for otherwise we can replace a vertex in $(R\cap B)-\{v\}$ to get a v-stable set R' of the size $\lfloor\frac{n}{k}\rfloor$, but $|R'\cap B|<|R\cap B|$, contradicting the choice of R. Therefore, any vertex of L is either in R or adjacent to some vertex in R. Then $(B\cup L)-R$ is a stable set of size at least $b+(a-b+1)-\lfloor\frac{n}{k}\rfloor\geq\lceil\frac{n}{k}\rceil$. Again, we are able to equitably k-color F. ∎

Notice that while it is easy to characterize equitable 2-colorability of a tree, it is slightly complicated for a forest. Suppose a forest F of order n has r components, each has order $n_i = a_i + b_i$ where a_i and b_i are the sizes of its partite sets. To check the equitable 2-colorability of F is the same as to partition $\{1, 2, \ldots, r\}$ into I and J such that $\sum_{i\in I} a_i + \sum_{j\in J} b_j = \lfloor \frac{n}{2} \rfloor$.

We close this note by raising the problem that how far can we go from trees to chordal graphs on equitable colorability.

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